Research Statement

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September 23, 2019

1 Introduction

Most broadly, I am interested in finding efficient combinatorial algorithms to compute information about toplogical and geometric spaces using finite descriptions. Since we live in an era of ubiquitous, powerful computers, questions about the computational aspects of geometry and topology are of fundamental interest. What information about a shape can we actually compute and therefore access and understand?

I am particularly interested in computing invariants and properties of low dimensional objects, particularly surfaces and 3-manifolds. These beautiful objects have the virtue of being the most picturable, and many interesting questions about them remain. My thesis is the first stage of a program to find a new combinatorial description and method for computing an important invariant of 3-manifolds called Heegaard Floer homology.

2 Heegaard Floer Homology

Heegaard Floer homology is an invariant of closed, orientable 3-manifolds defined by Ozsvath and Szabo in [1], defined using decompositions of 3-manifolds called Heegaard diagrams. Every closed, orientable 3-manifold M admits a Heegaard splitting: a representation of M as two handlebodies with homeomorphic boundary surfaces, and a choice of homeomorphism between the two boundary surfaces. A Heegaard diagram is a way of specifying a Heegaard splitting. It is a closed surface H of genus g marked with $2g$ closed curves, g of which are colored red and g of which are colored black. These are the attaching circles for the two handlebodies which join to make the 3-manifold M.

One way of constructing Heegaard Floer homolgy is to take the g-fold symmetric product of H with itself: that is, the space of unordered g-tuples of points in H. The red and black attaching circles generate corresponding red and black tori in the symmetric product $Sym_g(H)$. The chain groups are generated by g-tuples of intersection points of red attaching circles with black ones. The boundary map is then computed by giving a complex structure to H, which then induces one on $Sym_g(H)$.

In order to compute the Heegaard Floer boundary map, one counts the number of holomorphic representatives, up to a natural equivalence, of the following kind of homotopy class of maps. Let B be a bigon: that is, the unit disk in the complex plane with 2 marked points a and b which are the endpoints of a diameter. Call B a Whitney disk connecting g-tuples of intersection points x and y in $Sym_g(H)$ if B comes with a map f such that $f(a) = x$, $f(b) = y$, and f sends one arc on the boundary of B between a and b to the red torus in $Sym_g(H)$, and the other arc between a and b to the black torus. Under the right transversality and dimensionality conditions, the set of holormorphic representatives of a Whitney disk connecting x and y is finite. The boundary map of a chain x is computed by taking a sum over all chains y weighted by the number of holomorphic Whitney disks connecting x to y.

Whitney disks can be represented purely in terms of maps between surfaces. A lemma due to Oszvath and Szabo states that every holomorphic Whitney disk can be represented as a pair of maps from a Riemann surface with boundary $\widehat{D}$. The boundary of $\widehat{D}$ is marked with $2g$ vertices. Call the set of these vertices V. The components of $\partial\Sigma\backslash V$ are 2-colored red and black. One map is a degree-g holomorphic branched cover of the bigon B which respects the 2-coloring. The other is a holomorphic map $\phi : \widehat{D} \to H$ which sends components of $\partial\Sigma\backslash V$ to paths along attaching circles of the corresponding color, and which sends vertices to intersection points of attaching circles.

This formulation of a Whitney disk is the one I plan to use in order to build a new combinatorial description of Heegaard Floer Homology. Such a map as ϕ is determined up to homotopy by where it sends $\partial\widehat{D}$. ϕ may double back on a component of $\partial\Sigma\backslash V$. For each doubling back of this kind, we obtain a 1 parameter family of Whitney disks by varying the point where the image turns back on the attaching circle. The k-parameter set of Whitney disks generated by k doublings back is called a moving family.

Suppose $\widehat{D}$ has n boundary components. Counting the holomorphic representatives of a homotopy class of Whitney disks amounts to counting the intersection number of a moving family of dimension $2g + n - 1$ with the set of holomorphic Whitney disks. That set has codimension $2g + n - 1$ in the space of all Whitney disks, so when the right transversality conditions are met we expect the resulting intersection number to be finite. For this to work, one needs that the set of holmorphic Whitney disks sits within the space of Whitney disks much like an embedded submanifold does inside its ambient manifold. Ozsvath and Szabo prove this in [1].

3 Arc Diagrams and Degeneration

In order to make a new description of Heegaard Floer homology, I study a class of embedded graphs on surfaces with boundary, called weighted arc diagrams. These are graphs with 2-colored vertices located on the boundary of the surface in which they are embedded, and with nonnegative real wights assigned to each edge. The edges embed into the surface as homotopy classes (rel vertices) of paths connecting vertices which are mutually disjoint in the interior.

A toplogical branched covering map between surfaces S and S' defines a map from arc diagrams on S' to arc diagrams on S via path lifting. Weighted arc diagrams behave, in some contexts, much like a complex structure does. In fact, they represent what my advisor, Jeremy Kahn, and I call a degnenerate complex structure. A degenerate complex structure is a kind of generalization of a surface equipped with a quadratic differential. It is a structure which pulls back under branched covering maps, and can, on a surface with boundary, be represented by a weighted arc diagram.

To see how a quadratic differential on a surface Σ with boundary corresponds to a weighted arc diagram, first represent the quaratic differential with a measured foliation $\mathcal{F}$. By marking arcs on the boundary of Σ, we can compute the extremal length of the set of leaves of the foliation connecting marked arcs on the boundary of Σ. By replacing that whole family of leaves with a single arc and weighting that arc with the extremal length of those leaves, one can construct a weighted arc diagram from a quadratic differential.

One can take this notion further and, true to its name, use a quadratic differential to represent a degenerating path of complex structures. A quadratic differential on a surface Σ with boundary marks a geodesic in the Teichmüller space of Σ through contracting along the horizontal leaves of the foliation. By taking the projectivized limit of the extremal lengths of path families connecting boundary arcs, we obtain a limiting arc diagram which can be regarded as *the* degenerate complex structure of that surface.

I hope to replace the complex structure required in the original definition of a Whitney disk with a degnerate complex structure encoded by an arc diagram. Instead of putting a complex structure on the Heegaard surface H and pulling it back under a moving family of maps to the surface with red-black marked boundary Σ, we can instead put a quadratic differential on H. We can pull back the quadratic differential under each member of the moving family. We can then degenerate along the horizontal foliation of the quadratic differential and, taking the projective limit of extremal lengths of path families connecting labeled arcs, translate that quadratic differential into a weighted arc diagrams with 2-colored vertices. Arc diagrams are easy to work with from a computation standpoint, and this approach has the potential advantage of avoiding the messy perturbation theory required in the original formulation of Heegaard Floer homology.

The set of arc diagrams on a given surface Σ with n marked points on its boundary carries the structure of a finite dimenional simplicial complex, called the arc complex $\mathcal{A}(\Sigma)$ of Σ. Each arc on Σ is a vertex of $\mathcal{A}(\Sigma)$. A set of vertices span a simplex if we can choose representative paths which don't cross in the interior of Σ. If Σ has Euler characteristic $\chi(\Sigma)$, then at most $N = n - 3\chi(\Sigma)$ arcs can be mutually disjoint on Σ, so the top dimensional cells of the arc complex have dimension N.

A projective weighted arc diagram is one with its weights normalized to sum to 1. A projective wighted arc diagram can be represented by a point in barycentric coordinates on a simplex of $\mathcal{A}(\Sigma)$. $\mathcal{A}(\Sigma)$ geometrized by taking barycentric coordinates on each vertex is therefore called the projective weighted arc complex $\mathcal{PWA}(\Sigma)$.

In my thesis I studied membership in the set of projective weighted arc diagrams realizable by lifting a projective weighted arc diagram on a bigon, which we call the perfect locus $\mathcal{P}$. Membership in $\mathcal{P}$ is decidable. For maximal diagrams, that is, arc diagrams which cannot be contained in a larger arc diagram, I constructed an alogrithm which decides membership and proved that it is complete and correct. This algorithm is efficient and computationally tractable. I am still studying the computational complexity of the general membership problem for a non-maximal diagram.

The next stage in this program is to understand how $\mathcal{P}$ fits inside $\mathcal{PWA}(\Sigma)$. $\mathcal{P}$ is a simplicial complex embedded inside $\mathcal{PWA}(\Sigma)$. Its dimension is $N_{\mathcal{P}} = n/2 - \chi(\Sigma)$. We believe that $\mathcal{P}$ embeds into $\mathcal{PWA}$ similarly to the way a submanifold embeds inside a manifold. Concretely, we conjecture that the cohomology group $H^{N-N_{\mathcal{P}}}(\mathcal{PWA}, \mathcal{PWA}\backslash\mathcal{P}) = \mathbb{Z}$.

If that conjecture holds, then I could potentially construct a generator π of $H^{N-N_{\mathcal{P}}}(\mathcal{PWA}, \mathcal{PWA}\backslash\mathcal{P})$ which would compute the intersection number of $\mathcal{P}$ with a simplex of $\mathcal{PWA}$. In that case, much of the machinery in Heegaard Floer homology can be adapted to this new, degenerate setting. Instead of endowing the Heegaard surface with a complex structure, one could instead endow it with a quadratic differential. That quadratic differential would, through pulling back under the maps of a moving family of dimension $N - N_{\mathcal{P}}$ and then degenerating, yield a chain S in $\mathcal{PWA}$ of dimension $N - N_{\mathcal{P}}$ Whose boundary would generically lie in $\mathcal{PWA}\backslash\mathcal{P}$. Its intersection number with $\mathcal{P}$ could then be computed using $\pi(S)$. This would effectively count the number of holomorphic degenerate Whitney disks.

By thinking of arc diagrams as degenerate complex structures, the complex of projective weighted arc diagrams on a surface and the perfect locus inside it are very close analogs of the space of Whitney disks and its subset of holomorphic representatives. This is what motivates the conjecture that $H^{N-N_{\mathcal{P}}}(\mathcal{PWA}, \mathcal{PWA}\backslash\mathcal{P}) = \mathbb{Z}$. It would be of its own independent interest if this conjecture bears out.

The final stage of the program would be to do this adaptation and, hopefully, obtain a procedure implementable on a computer to compute the Heegaard Floer homology of closed oriented 3-manifold using operations on weighted arc diagrams. That membership in $\mathcal{P}$ is computable, efficiently so in the maximal case, is a promising sign. The current phase of the program is now to prove the conjecture, and come to grips with how the set of perfect diagrams fits inside the projective weighted arc complex.

References

[1] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and topological invariants for closed three-manifolds. *Annals of Mathematics*, 159(3):1027–1158, 2004.